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Predictive Functional Control Based on Fuzzy Model: Design and Stability Study

IGOR ŠKRJANC and SAŠO BLAŽIČ
Faculty of Electrical Engineering, University of Ljubljana, Tržaška 25, 1000 Ljubljana, Slovenia; e-mail: igor.skrjanc@fe.uni-lj.si

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Abstract. In the paper the design methodology and stability analysis of parallel distributed fuzzy model based predictive control is presented. The idea is to design a control law for each rule of the fuzzy model and blend them together. The proposed control algorithm is developed in state space domain and is given in analytical form. The analytical form brings advantages in comparison with optimization based control schemes especially in the sence of realization in real-time. The stability analysis and design problems can be viewed as a linear matrix inequalities problem. This problem is solved by convex programming involving LMIs. In the paper a sufficient stability condition for parallel distributed fuzzy model-based predictive control is given. The problem is illustrated by an example on magnetic suspension system.

Key words: fuzzy identification, predictive control, stability.

1. Introduction

The controlled processes are inherently nonlinear in nature, but the majority of model-based predictive control (MBPC) applications and algorithms up to date are based on linear models. The reason for that is in the use of a linear model and a quadratic objective function where the nominal MBPC algorithms take the form of a highly structured convex quadratic program, for which the solution can be easily found.

In some highly nonlinear cases the use of nonlinear model-based predictive control (NMBPC) can be easily justified. By introducing the nonlinear model into predictive control problem, the complexity increases significantly. In [1] and [5] an overview of different nonlinear predictive control approaches is given and discussed.

Many approaches to the nonlinear predictive control have been proposed in recent years. They differ according to the predictive algorithm and according to the model which approximates the process dynamics [3, 4, 7–11]. Fuzzy model-based predictive control has become particularly important in recent years.

The main focus in the paper is given to predictive functional control based on fuzzy model which is a parallel distributed fuzzy model-based predictive control algorithm (PFCFM). The PFCFM is actually a collection of individual predictive control laws for each fuzzy model rule, which are blended together at the end. The control law in the case of PFCFM approach is based on fuzzy model and is given in analytical form. This makes the approach very easy for implementation in real-time applications.

The most important part of the paper is stability study of the proposed control system. The investigation of the stability in the case of nonlinear control systems which are based on fuzzy models is especially difficult and important task. The problem which arises is that the fuzzy model which has locally asymptotically stable subsystems, can be globally unstable [16]. The problem of PFCFM stability analysis is, in general, impossible to be solved in an analytical manner. These problems have been approached by linear matrix inequalities (LMIs) in recent years [16]. The stability problem of fuzzy systems can be viewed as an LMI problem. Instead of developing an analytical solution, the problem is reformulated to verification whether an LMI is solvable by optimizing functionals over LMI constraints. In the paper a sufficient stability condition for parallel distributed fuzzy model-based predictive control is given.

The paper is organized in the following way: in Section 2 the parallel distributed fuzzy model-based predictive control is presented, Section 3 deals with stability analysis of PFCFM using LMIs and in Section 4 the design and stability analysis is given for magnetic suspension system.

2. Parallel Distributed Fuzzy Model-based Predictive Control

The fuzzy model is described by rules [12] which locally describe linear input output relations of the system in state space form:

$$\begin{aligned}\mathbf{R}_j:\quad & \text{if } x_{p1}(k) \text{ is } \mathbf{P}_{1,k_1} \text{ and } x_{p2}(k) \text{ is } \mathbf{P}_{2,k_2} \text{ and } \dots \text{ and } x_{pq}(k) \text{ is } \mathbf{P}_{q,k_q}\\ & \text{then } x_m(k+1) = \mathbf{A}_{m,j}\mathbf{x}_m(k) + \mathbf{B}_{m,j}u(k-d_j), \quad j = 1, \dots, m,\\ & \quad k_1 = 1, \dots, f_1,\ k_2 = 1, \dots, f_2,\ \dots,\ k_q = 1, \dots, f_q.\end{aligned} \tag{1}$$

The q-element vector $\mathbf{x}_p^{\mathrm{T}}(k) = [x_{p1}(k), \dots, x_{pq}(k)]$ denotes the input or variables in premise, and variable y is the output of the model. With each variable in premise $x_{pi}(k)$ $(i = 1, \dots, q)$, f_i fuzzy sets $(\mathbf{P}_{i,1}, \dots, \mathbf{P}_{i,f_i})$ are connected, and each fuzzy set $\mathbf{P}_{i,k_i}$ $(k_i = 1, \dots, f_i)$ is associated with a real-valued function $\mu_{P_{i,k_i}}(x_{pi})$: $\mathbb{R} \to [0, 1]$, that produces membership grade of the variable x_{pi} with respect to the fuzzy set $\mathbf{P}_{i,k_i}$. $\mathbf{A}_{m,j}$ and $\mathbf{B}_{m,j}$ for $j = 1, \dots, m$ are system model state-space matrices and assuming also the time delay d_j, where $d_j = \mathit{round}(T_j/T_{\mathrm{s}})$ and T_j stands for time delay in jth operating domain, and T_{s} for sampling time. To make the list of fuzzy rules complete, all possible variations of fuzzy sets are given in (1), yielding the number of fuzzy rules $m = f_1 \times f_2 \times \cdots \times f_q$. The variables x_{pi} are not the only inputs of the fuzzy system. Implicitly, the n-element vector $x_m(k)^{\mathrm{T}} = [x_{m,1}(k), \dots, x_{m,n}(k)]$ also represents the input to the system. It is usually referred to as the consequence vector.

The whole output of the system is given by the following equation:

$$\mathbf{x}_m(k+1) = \frac{\sum_{k_1=1}^{f_1}\sum_{k_2=1}^{f_2}\cdots\sum_{k_q=1}^{f_q}\mu_{P_{1,k_1}}(x_{p1})\mu_{P_{2,k_2}}(x_{p2})\cdots\mu_{P_{q,k_q}}(x_{pq})}{\sum_{k_1=1}^{f_1}\sum_{k_2=1}^{f_2}\cdots\sum_{k_q=1}^{f_q}\mu_{P_{1,k_1}}(x_{p1})\mu_{P_{2,k_2}}(x_{p2})\cdots\mu_{P_{q,k_q}}(x_{pq})} \times (\mathbf{A}_{m,j}\mathbf{x}_m(k) + \mathbf{B}_{m,j}u(k-d_j)). \tag{2}$$

To simplify (2), a partition of unity is considered where functions $\beta_j(\mathbf{x}_p)$ defined by

$$\beta_j(\mathbf{x}_p) = \frac{\mu_{P_{1,k_1}}(x_{p1})\mu_{P_{2,k_2}}(x_{p2})\cdots\mu_{P_{q,k_q}}(x_{pq})}{\sum_{k_1=1}^{f_1}\sum_{k_2=1}^{f_2}\cdots\sum_{k_q=1}^{f_q}\mu_{P_{1,k_1}}(x_{p1})\mu_{P_{2,k_2}}(x_{p2})\cdots\mu_{P_{q,k_q}}(x_{pq})},$$
$$j = 1,\ldots,m, \text{ while } k_1 = 1,\ldots,f_1, \ \ldots, \ k_q = 1,\ldots,f_q, \tag{3}$$

give information about the fulfilment of the respective fuzzy rule in the normalized form. It is obvious that $\sum_{j=1}^{m}\beta_j(\mathbf{x}_p) = 1$ irrespective of $\mathbf{x}_p$ as long as the denominator of $\beta_j(\mathbf{x}_p)$ is not equal to zero (that can be easily prevented by stretching the membership functions over the whole potential area of $\mathbf{x}_p$).

Combining (2) and (3) and changing summation over k_i by summation over j we arrive to the following fuzzy model in state-space domain:

$$\mathbf{x}_m(k+1) = \sum_{j=1}^{m}\beta_j(\mathbf{x}_p)(\mathbf{A}_{m,j}\mathbf{x}_m(k) + \mathbf{B}_{m,j}u(k-d_j)), \tag{4}$$

$$y_m(k) = \sum_{j=1}^{m}\beta_j(\mathbf{x}_p)\mathbf{C}_{m,j}\mathbf{x}_m(k). \tag{5}$$

The use of membership functions in input space with overlapping receptive fields provides interpolation and extrapolation and it has been shown in [6, 15, 17] that fuzzy models can be viewed as universal approximators.

The problem of delays in the plant is circumvented by constructing an auxiliary variable that serves as the output of the plant if there were no delay present. The so-called “undelayed” model of the plant will be introduced for that purpose. It is obtained by “removing” delay from the “delayed” model:

$$\mathbf{x}_m^0(k+1) = \sum_{j=1}^{m}\beta_j(\mathbf{x}_p)(\mathbf{A}_{m,j}\mathbf{x}_m^0(k) + \mathbf{B}_{m,j}u(k)), \tag{6}$$

$$y_m^0(k) = \sum_{j=1}^{m}\beta_j(\mathbf{x}_p)\mathbf{C}_{m,j}\mathbf{x}_m^0(k), \tag{7}$$

where $y_m^0(k)$ models the “undelayed” output of the plant.

The fuzzy model described by Equations (6) and (7) has fixed real matrices $\mathbf{A}_{m,j}$, $j = 1, \ldots, m$, of the dimensions $n \times n$, $B_{m,j}$, $j = 1, \ldots, m$, of the dimensions $n \times 1$ and $\mathbf{C}_{m,j}$, $j = 1, \ldots, m$, of the dimension $1 \times n$. Those matrices are interpreted as coefficients of a convex decomposition of the time-varying or nonlinear system matrices. We refer to such a model as a *polytopic model*. No special restrictions are imposed to those matrices. However, as it will be shown in the following, the integral action of the proposed control is achieved when all local systems are in the controllable canonical form. It is useful to introduce new matrices:

$$
\begin{aligned}
\overline{\mathbf{A}}_m &= \sum_j \beta_j(\mathbf{x}_p(k))\mathbf{A}_{m,j},\\
\overline{\mathbf{B}}_m &= \sum_j \beta_j(\mathbf{x}_p(k))\mathbf{B}_{m,j},\\
\overline{\mathbf{C}}_m &= \sum_j \beta_j(\mathbf{x}_p(k))\mathbf{C}_{m,j}.
\end{aligned}
\tag{8}
$$

It is obvious that these matrices are not constant. Rather, they change as system changes its operating point. Introducing Equation (8) into Equations (6) and (7) we obtain:

$$
\begin{aligned}
\mathbf{x}_m^0(k+1) &= \overline{\mathbf{A}}\mathbf{x}_m^0(k) + \overline{\mathbf{B}}u(k),\\
y_m^0(k) &= \overline{\mathbf{C}}\mathbf{x}_m^0(k).
\end{aligned}
\tag{9}
$$

The model form (9) is used for fuzzy model-based predictive control design. The behavior of the closed-loop system in the case of almost all predictive control techniques is defined by the reference trajectory which is given implicitly or explicitly. The control goal is to determine the future control action so that the predicted output value coincides with the reference trajectory. The point where the reference and output signals coincide is called a coincidence horizon and is denoted by H. The prediction is calculated under assumption of constant future manipulated variables ($u(k) = u(k+1) = \cdots = u(k+H-1)$), i.e. the mean level control and under assumption of constant β_j, $j = 1, \ldots, m$, through the whole prediction horizon. Under those assumption the H-step ahead prediction of the "undelayed" plant output at time instant k is obtained as

$$
y_m^0(k+H|k) = \overline{\mathbf{C}}_m\big(\overline{\mathbf{A}}_m^H\mathbf{x}_m^0(k) + (\overline{\mathbf{A}}_m^H - \mathbf{I})(\overline{\mathbf{A}}_m - \mathbf{I})^{-1}\overline{\mathbf{B}}_m u(k)\big).
\tag{10}
$$

The reference-model trajectory is given implicitly by exponential factor which describes how the control error should behave in future. Through this exponential factor a_r, which is analog to the time constant of the reference model, we will predict the control error H-step ahead. It has to be pointed out that our approach does not involve explicit coincidence of the reference model and the plant output. It only demands the exponential decreasing of the control error as given next:

$$
w(k+H|k) - y_p^0(k+H|k) = a_r^H \cdot (w(k) - y_p^0(k)),
\tag{11}
$$

where $w(k)$ and $y_p^0(k)$ stand for the current reference signal and output signal of the "undelayed" plant, respectively. Assuming constant reference signal in the future, the reference trajectory is given as follows:

$$y_p^0(k+H|k) = w(k) - a_r^H \cdot (w(k) - y_p^0(k)). \quad (12)$$

The main goal of the proposed algorithm is to find the control law which enables the reference trajectory tracking of the "undelayed" controlled signal. In other words, $u(k)$ has to be found to fulfil (12). The estimated value of the "undelayed" process output is given as

$$y_p^0(k+H|k) = y_p^0(k) + y_m^0(k+H|k) - y_m^0(k). \quad (13)$$

It is obtained under assumption that the plant output will change for the same amount as its model in the same interval of time.

Combining (12), (13) and the prediction of model output given in (10), the following is obtained:

$$u(k) = g_0^{-1}\big((1-a_r^H)(w(k) - y_p^0(k)) + \overline{\mathbf{C}}_m(\mathbf{I} - \overline{\mathbf{A}}_m^H)\mathbf{x}_m^0(k)\big), \quad (14)$$

where g_0 stands for

$$g_0 = \overline{\mathbf{C}}_m(\overline{\mathbf{A}}_m^H - \mathbf{I})(\overline{\mathbf{A}}_m - \mathbf{I})^{-1}\overline{\mathbf{B}}_m. \quad (15)$$

The variable $y_p^0(k)$ cannot be measured directly. Rather, it will be estimated from the available signals:

$$y_p^0(k) = y_p(k) - y_m(k) + y_m^0(k). \quad (16)$$

Equation (16) is obtained by the same reasoning as Equation (13). Including (16) in (14), the control law of PFCFM in analytical form is obtained:

$$\begin{aligned} u(k) \; &= \; g_0^{-1}\big((1-a_r^H)(w(k) - y_p(k) + y_m(k) - y_m^0(k)) \\ &\quad + \overline{\mathbf{C}}_m(\mathbf{I} - \overline{\mathbf{A}}_m^H)\mathbf{x}_m^0(k)\big). \end{aligned} \quad (17)$$

Note that the control law (17) is realizable if the gain g_0 is nonzero. This is true if $H \geqslant \rho$, where ρ is the relative order of the system (the difference between the number of system poles and system zeros).

The parameters of the control law (17) are time-varying. This is the main drawback of the control law (17), because it is difficult to prove the stability when the controller parameters are time-varying. This was the reason to transform the proposed control law to parallel distributed control law. This means that the control law is defined as m parallel constant controllers which are blended together to compose the whole control signal as given next:

$$\begin{aligned} u(k) \; &= \; \sum_{j=1}^{m} \beta_j\big(g_{0,j}^{-1}\big((1-a_r^H)(w(k) - y_p(k) + y_m(k) - y_m^0(k)) \\ &\quad + \mathbf{C}_{m,j}(\mathbf{I} - \mathbf{A}_{m,j}^H)\mathbf{x}_m^0(k)\big)\big) \end{aligned} \quad (18)$$

and

$$g_{0,j} = \mathbf{C}_{m,j}(\mathbf{A}_{m,j}^{H} - \mathbf{I})(\mathbf{A}_{m,j} - \mathbf{I})^{-1}\mathbf{B}_{m,j}. \tag{19}$$

In general, the above mentioned control laws (described by Equations (17) and (18)) are not equivalent. They are equal when only one membership value is activated, i.e. only one membership value is different from zero. However, even if this is not the case, the two control laws produce similar results, but the stability analysis is different.

Next, we will discuss the properties of proposed parallel distributed fuzzy model-based predictive control which is given by Equations (18) and (19).

The very important feature of all control algorithms is the nature at low frequencies. The control algorithm should be able to suppress the control error in steady-state, i.e. the control law should have the integral nature. When we discuss the behavior at steady-state, we can assume constant membership values β_i, $i = 1, \ldots, m$, and Z-transformation can be applied over (18):

$$\begin{aligned} U(z) &= \sum_{j=1}^{m} \beta_j \big(g_{0,j}^{-1}\big((1 - a_r^H)E(z) \\ &\quad + \mathbf{C}_{m,j}(\mathbf{I} - \mathbf{A}_{m,j}^{H})(z\mathbf{I} - \overline{\mathbf{A}}_m)^{-1}\overline{\mathbf{B}}_m U(z)\big)\big), \\ E(z) &= W(z) - Y_p(z) + Y_m(z) - Y_m^0(z) = W(z) - Y_p^0(z). \end{aligned} \tag{20}$$

Solving (20) for $U(z)$ yields

$$U(z) = (G_i(z))^{-1} \sum_{j=1}^{m} \beta_j \big(g_{0,j}^{-1}(1 - a_r^H)\big)E(z), \tag{21}$$

$$G_i(z) = 1 - \sum_{j=1}^{m} \beta_j \big(g_{0,j}^{-1}\mathbf{C}_{m,j}(\mathbf{I} - \mathbf{A}_{m,j}^{H})\big)(z\mathbf{I} - \overline{\mathbf{A}}_m)^{-1}\overline{\mathbf{B}}_m. \tag{22}$$

The integral nature of the proposed control is proven if it can be shown that the transfer function $U(z)/E(z)$ has infinite DC-gain. This is equivalent to showing that $G_i(z)|_{z=1} = 0$. It was mentioned before that the integral action is achieved if the "local" systems in all fuzzy domains are in the controllable canonical form:

$$\mathbf{A}_{m,j} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_{j,n} & -a_{j,n-1} & -a_{j,n-2} & \ldots & -a_{j,1} \end{bmatrix}, \qquad \mathbf{B}_{m,j} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{23}$$

$$\mathbf{C}_{m,j} = [\, b_{j,n} \quad b_{j,n-1} \quad \ldots \quad b_{j,1} \,]. \tag{24}$$

It is easy to show that

$$(\mathbf{I}-\mathbf{A}_{m,j})^{-1}\mathbf{B}_{m,j} = \frac{1}{1+\sum_{i=1}^{n} a_{j,i}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{25}$$

It is obvious that the matrices defined in Equation (8) are also in the controllable canonical form and similar expression as the one in (25) can be calculated as follows:

$$(\mathbf{I}-\overline{\mathbf{A}}_{m})^{-1}\overline{\mathbf{B}}_{m} = \frac{1}{1+\sum_{i=1}^{n}\sum_{j=1}^{m} \beta_j a_{j,i}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{26}$$

First, $g_{0,j}$ defined in Equation (19) will be simplified using Equation (25):

$$\begin{aligned} g_{0,j} &= \mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^{H})(\mathbf{I}-\mathbf{A}_{m,j})^{-1}\mathbf{B}_{m,j} \\ &= \mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^{H})\frac{1}{1+\sum_{i=1}^{n} a_{j,i}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = g_j^{*}\frac{1}{1+\sum_{i=1}^{n} a_{j,i}}, \end{aligned} \tag{27}$$

where g_j^{*} was introduced as

$$g_j^{*} = \mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^{H}) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{28}$$

The inverse of $g_{0,j}$ is needed in Equation (22) and it follows from Equation (27):

$$g_{0,j}^{-1} = g_j^{*-1}\left(1+\sum_{i=1}^{n} a_{j,i}\right). \tag{29}$$

Introducing Equations (29) and (26) into Equation (22) we obtain

$$\begin{aligned} G_i(z)|_{z=1} &= 1-\sum_{j=1}^{m}\beta_j\left(g_j^{*-1}\left(1+\sum_{i=1}^{n} a_{j,i}\right)\mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^{H})\right) \\ &\quad\times\frac{1}{1+\sum_{i=1}^{n}\sum_{j=1}^{m} \beta_j a_{j,i}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned} \tag{30}$$

Further introducing Equation (28) into Equation (30) we arrive to the following

$$\begin{aligned} G_i(z)|_{z=1} &= 1-\sum_{j=1}^{m}\beta_j\left(g_j^{*-1}\left(1+\sum_{i=1}^{n} a_{j,i}\right)g_j^{*}\right) \\ &\quad\times\frac{1}{1+\sum_{i=1}^{n}\sum_{j=1}^{m} \beta_j a_{j,i}} = 1-1=0, \end{aligned} \tag{31}$$

where the identity $\sum_{j=1}^{m} \beta_j = 1$ was taken into account. It was shown in Equation (31) that the DC-gain of the transfer function $U(z)/E(z)$ is indeed infinite, proving the integral property of the controller.

3. Stability Analysis Using LMI

The stability of control systems is one of their most important features. The investigation of the stability in the case of the control systems which are based on fuzzy models is especially difficult and important task, because we are generally dealing with nonlinear plants. The problem which arises is that the fuzzy model which has locally asymptotically stable subsystems, i.e. all submatrices are Hurwitz, can be globally unstable [16].

The problem of PFCFM stability analysis is in general impossible to be solved in an analytical manner. These problems have been approached by linear matrix inequalities (LMIs) in recent years [16]. The stability problem of fuzzy systems can be viewed as an LMI problem. Instead of developing an analytical solution, the problem is solved numerically by reformulating it to verification whether an LMI is solvable, i.e. optimizing functionals over LMI constraints.

3.1. LINEAR MATRIX INEQUALITY

To introduce linear matrix inequalities [2], the mapping **F** is formulated as

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + x_2\mathbf{F}_2 + \cdots + x_m\mathbf{F}_m, \tag{32}$$

where $\mathbf{x}^{\mathrm{T}} = [x_1, x_2, \ldots, x_m]$ is a vector of m real values which are called the *decision variables*, $\mathbf{F}_0, \ldots, \mathbf{F}_m$ are real symmetric matrices ($\mathbf{F}_i = \mathbf{F}_i^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $i = 0, \ldots, m$ for $n \in \mathbb{Z}_+$).

DEFINITION 1. A *Linear Matrix Inequality* is an inequality

$$\mathbf{F}(\mathbf{x}) \succ 0, \tag{33}$$

where **F** is an affine function mapping of a vector space $\mathbb{V}$ to the set $\mathbb{S} = \{\mathbf{M} \mid \exists n > 0 \text{ such that } \mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \mathbb{R}^{n\times n}\}$ as defined in (32).

Note that '$\succ$' in (33) denotes that $\mathbf{F}(\mathbf{x})$ is positive definite, i.e., $\mathbf{u}^{\mathrm{T}}\mathbf{F}(\mathbf{x})\mathbf{u} > 0$, $\forall \mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} \neq 0$. Equivalently, this means that the smallest eigenvalue of $\mathbf{F}(\mathbf{x})$ is positive.

Remark 1. An example of LMIs is Lyapunov inequality $\mathbf{F}(\mathbf{x}) = \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} - \mathbf{P} + \mathbf{Q} \prec 0$, where **F** applies an affine mapping $\mathbf{F}\colon \mathbb{R}^{n\times n} \to \mathbb{S}$. The matrices $\mathbf{A}, \mathbf{Q} \in \mathbb{R}^{n\times n}$ are assumed to be known and $\mathbf{P} \in \mathbb{R}^{n\times n}$ is the unknown matrix. The Lyapunov inequality defines an LMI only if the matrix $\mathbf{Q}$ is symmetric. In this case the Lyapunov inequality can be rewritten in the form as given in (32):

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}\left(\sum_{j=1}^{m} x_j \mathbf{E}_j\right) = \mathbf{F}_0 + \sum_{j=1}^{m} x_j \mathbf{F}(\mathbf{E}_j) = \mathbf{F}_0 + \sum_{j=1}^{m} x_j \mathbf{F}_j, \tag{34}$$

where $\mathbf{E}_1, \ldots, \mathbf{E}_m$ define the basis of $\mathbb{V}$ and $\mathbf{x}$ can be defined as $\mathbf{x} = \sum_{j=1}^{m} x_j \mathbf{E}_j$.

DEFINITION 2. *A system of linear matrix inequalities* is a finite set of linear matrix inequalities

$$\mathbf{F}_1(\mathbf{x}) \succ 0, \quad \mathbf{F}_2(\mathbf{x}) \succ 0, \quad \ldots, \quad \mathbf{F}_m(\mathbf{x}) \succ 0. \tag{35}$$

A very important property of the system of LMIs is that it can be rewritten as a single LMI. This means that the inequalities in (35) are valid if and only if

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} \mathbf{F}_1(\mathbf{x}) & 0 & \ldots & 0 \\ 0 & \mathbf{F}_2(\mathbf{x}) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathbf{F}_m(\mathbf{x}) \end{bmatrix} \succ 0. \tag{36}$$

The inequality (36) makes sense only when $\mathbf{F}(\mathbf{x})$ is symmetric and in that case the eigenvalues of matrix are simply the union of the eigenvalues of $\mathbf{F}_1(\mathbf{x}), \ldots, \mathbf{F}_m(\mathbf{x})$.

3.2. STABILITY ANALYSIS OF PFCFM USING LMI

The stability analysis of the proposed PFCFM control can be performed using an LMI approach shown in [16] and [14] assuming an ideal model of the plant is available ($y_p(k) = y_m(k)$, $k \in \mathbb{Z}_+$). The stability problem of PFCFM is reduced to the stability analysis of autonomous closed-loop system ($w(k) \equiv 0$, $k \in \mathbb{Z}_+$), which is obtained when the predictive control law from (18) is simplified due to the above given assumptions and introducing Equation (7) which yields the following:

$$u(k) = \sum_{j=1}^{m} \beta_j g_{0,j}^{-1}\left((a_r^H - 1)\sum_{l=1}^{m} \mathbf{C}_{m,l}\mathbf{x}_m^0(k) + \mathbf{C}_{m,j}(\mathbf{I} - \mathbf{A}_{m,j}^H)\mathbf{x}_m^0(k)\right). \tag{37}$$

When the control action (37) is introduced into Equation (6), we obtain

$$\begin{aligned} \mathbf{x}_m^0(k+1) = {} & \sum_{i=1}^{m} \beta_i \Bigg(\mathbf{A}_{m,i} + \mathbf{B}_{m,i}\sum_{j=1}^{m} \beta_j g_{0,j}^{-1}\Bigg((a_r^H - 1)\sum_{l=1}^{m} \beta_l \mathbf{C}_{m,l} \\ & + \mathbf{C}_{m,j}(\mathbf{I} - \mathbf{A}_{m,j}^H)\Bigg)\Bigg)\mathbf{x}_m^0(k) \end{aligned} \tag{38}$$

which can be written in a way that is more suitable for further analysis:

$$\begin{aligned}\mathbf{x}_m^0(k+1) \;=\; & \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{l=1}^{m}\beta_i\beta_j\beta_l\big(\mathbf{A}_{m,i}+\mathbf{B}_{m,i}g_{0,j}^{-1}(\mathbf{C}_{m,l}(a_r^H-1)\\ & +\mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^H))\big)\mathbf{x}_m^0(k).\end{aligned} \tag{39}$$

In the case when matrices $\mathbf{C}_{m,j}$ are the same in all fuzzy domains (this happens when the zeros of the discrete transfer functions are the same) and can be denoted by $\mathbf{C}_m$, the stability analysis is simpler since summation over l is no longer needed in Equation (39) and the system can be described by

$$\mathbf{x}_m^0(k+1) = \sum_{i=1}^{m}\sum_{j=1}^{m}\beta_i\beta_j\big(\mathbf{A}_{m,i}+\mathbf{B}_{m,i}g_{0,j}^{-1}\mathbf{C}_m(a_r^H\mathbf{I}-\mathbf{A}_{m,j}^H)\big)\mathbf{x}_m^0(k). \tag{40}$$

The stability condition for fuzzy systems using Lyapunov approach is given in [13]. The following theorem adapted from [13] gives sufficient condition for global asymptotical stability of a fuzzy system using LMIs:

The fuzzy system given in the following form

$$\mathbf{x}_m^0(k+1) = \sum_{j=1}^{m}\beta_j\mathbf{A}_{c,j}\mathbf{x}_m^0(k) \tag{41}$$

is global asymptotically stable if there exist a common positive definite matrix $\mathbf{P}$, $\mathbf{P}=\mathbf{P}^{\mathrm{T}}$, *that fulfils the set of inequalities:*

$$\mathbf{A}_{c,j}^{\mathrm{T}}\mathbf{P}\mathbf{A}_{c,j}-\mathbf{P}\prec 0, \quad j=1,\ldots,m. \tag{42}$$

Proof. This means that the Lyapunov function $V(\mathbf{x}(k)) = \mathbf{x}^{\mathrm{T}}(k)\mathbf{P}\mathbf{x}(k) \succ 0$, $\forall\mathbf{x}(k)$ and $V(\mathbf{0})=0$, $\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \prec 0$, $\forall\mathbf{x}(k)$, $\Delta V(\mathbf{0})=0$ and $\lim_{\|\mathbf{x}(k)\|\to\infty} V(\mathbf{x}(k)) = \infty$. □

As it has been stated, the fuzzy system which is locally asymptotically stable, i.e. all system submatrices are Hurwitz, can be globally unstable [16]. This means that no common $\mathbf{P}$ can be found to fulfill (42). So, it is not sufficient to check the eigenvalues of system submatrices.

The problem of global asymptotical stability in the case of PFCFM is reformulated to verification of LMI feasibility, i.e. the PFCFM system is stable if there exist a common positive definite matrix $\mathbf{P}$ such that

$$(\mathbf{A}_{m,i}+\mathbf{B}_{m,i}\mathbf{K}_{j,l})^{\mathrm{T}}\mathbf{P}(\mathbf{A}_{m,i}+\mathbf{B}_{m,i}\mathbf{K}_{j,l})-\mathbf{P}\prec 0, \quad i,j,l=1,\ldots,m, \tag{43}$$

where $\mathbf{K}_{j,l}$ stands for

$$\mathbf{K}_{j,l} = g_{0,j}^{-1}(\mathbf{C}_{m,l}(a_r^H-1)+\mathbf{C}_{m,j}(\mathbf{I}-\mathbf{A}_{m,j}^H)), \tag{44}$$

$$g_{0,j} = \mathbf{C}_m(\mathbf{A}_{m,j}^H-\mathbf{I})(\mathbf{A}_{m,j}-\mathbf{I})^{-1}\mathbf{B}_{m,j}. \tag{45}$$

The system of LMIs (43) can be reformulated in one LMI. In order to do this, the matrices $A_{c,ijl}$ are introduced as

$$\mathbf{A}_{c,ijl} = \mathbf{A}_{m,i} + \mathbf{B}_{m,i}\mathbf{K}_{j,l}, \quad i, j, l = 1, \ldots, m. \tag{46}$$

It has to be noted that investigation of PFCFM stability only needs to involve those combinations of i, j, and l that can occur, i.e. only overlapping membership functions have to be analysed. The global asymptotical stability is obtained when the following condition is fulfilled:

$$\mathbf{A}_c^{\mathrm{T}}\mathbf{P}_c\mathbf{A}_c - \mathbf{P}_c \prec 0, \tag{47}$$

where

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{A}_{c,111} & 0 & \ldots & 0 \\ 0 & \mathbf{A}_{c,112} & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \mathbf{A}_{c,mmm} \end{bmatrix} \tag{48}$$

and $\mathbf{P}_c \succ 0$ is defined as follows:

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{P} & 0 & \ldots & 0 \\ 0 & \mathbf{P} & \vdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \mathbf{P} \end{bmatrix} \succ 0, \tag{49}$$

where $\mathbf{P}$ is an arbitrary symmetric positive definite matrix.

4. Design and Stability Analysis of PFCFM for Magnetic Suspension System

The design and stability study of PFCFM will be illustrated on the case of magnetic suspension system. The latter consists of an electromagnet, a coil and a distance sensor. In Figure 1 the basic principle where u_{RL} and i are the voltages and the

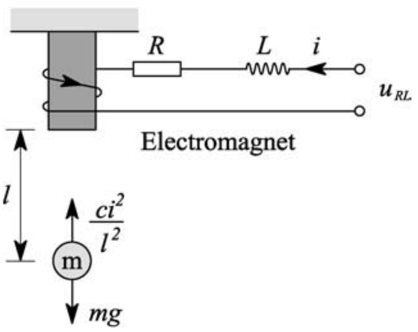

Figure 1. The magnetic suspension system.

current of the electromagnet, respectively, is shown. Parameters $R = 10.5\ \Omega$ and $L = 15.8$ mH are resistance and inductance of the electromagnet, respectively, $c = 2.72 \cdot 10^{-5}\ \mathrm{Nm^2\,A^{-2}}$, $m = 0.084$ kg is the mass of the coil and l is the distance between the electromagnet and the coil.

4.1. MODELLING OF THE PLANT

From the second Newton law we obtain

$$mg - F_m = m\frac{\mathrm{d}^2 l}{\mathrm{d}t^2}, \tag{50}$$

where the magnetic force F_m depends on the current i, the distance l and the parameter c:

$$F_m = c\frac{i^2}{l^2}. \tag{51}$$

The electrical part of the system is modelled by the following equation

$$u_{RL}(t) = L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri. \tag{52}$$

The system of nonlinear differential equations comprising of Equations (50), (51) and (52) can be seen as a nonlinear differential equation of third order. The system which is open-loop unstable is compensated and stabilized in large by lead compensator $5(s+40)/(s+400)$. The resulting closed-loop system was identified

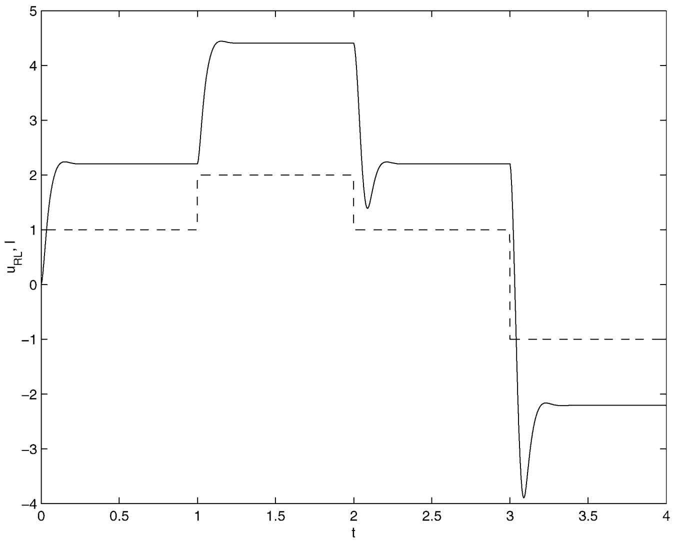

Figure 2. Dynamic response of the stabilized system: the input signal u_{RL} – dashed, the output l – solid.

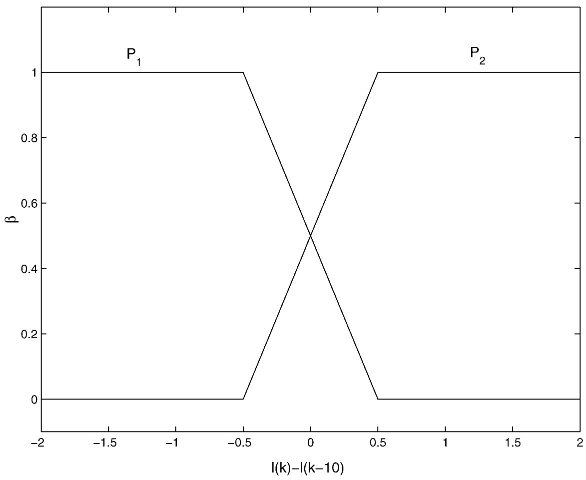

Figure 3. The membership functions.

using fuzzy modelling. The experiment was conducted and the results are shown in Figure 2. It is obvious that the dynamics of the responses depends most obviously on the direction of the controlled variable, i.e. the responses of the system are much different when the coil is going up from the case when it is going down. Due to the two main behaviours the structure with two rules and triangular shaped membership functions was chosen as shown in Figure 3.

The compensated system is modelled in a form of the third order discrete-time model with the variable in premise $\mathbf{x}_p^{\mathrm{T}}(k)$ which defines the different dynamics according to the trend of the distance $\mathbf{x}_p^{\mathrm{T}}(k) = [l(k) - l(k-10)]$ (the value 10 here was chosen heuristically) and the consequence vector as $\mathbf{x}^{\mathrm{T}}(k) = [l(k), l(k-1), l(k-2)]$. The structure of fuzzy model used for the model of the plant is given next:

$$\begin{aligned} \mathbf{R}_j\colon\quad & \text{if } l(k) - l(k-10) \text{ is } \mathbf{P}_j \text{ then } \mathbf{x}(k+1) = \mathbf{A}_{m,j}\mathbf{x}(k) + \mathbf{B}_{m,j}u(k), \\ & y_m(k) = \mathbf{C}_{m,j}\mathbf{x}(k), \quad j = 1, 2. \end{aligned} \tag{53}$$

Performing fuzzy identification of the model (53) the following matrices in controllable canonical form are obtained

$$\begin{aligned} \mathbf{A}_{m,1} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5822 & 0.2288 & 1.3449 \end{bmatrix}, \qquad \mathbf{B}_{m,1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \mathbf{C}_{m,1} &= [\,0 \quad 0 \quad 0.0188\,], \end{aligned} \tag{54}$$

$$\begin{aligned} \mathbf{A}_{m,2} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.4755 & 0.1495 & 1.3174 \end{bmatrix}, \qquad \mathbf{B}_{m,2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ \mathbf{C}_{m,2} &= [\,0 \quad 0 \quad 0.0188\,]. \end{aligned} \tag{55}$$

The system is with zeros in the origin and the relative degree is of the first order ($\rho = 1$). It is important that $\mathbf{C}_{m,1} = \mathbf{C}_{m,2}$ what simplifies the stability analysis.

4.2. DESIGN AND STABILITY ANALYSIS OF PFCFM FOR MAGNETIC SUSPENSION SYSTEM

The PFCFM control algorithm has two design parameters – the reference trajectory constant or the reference model pole a_r and horizon H. The proposed reference model pole was chosen as $a_r = 0.92$. The fuzzy model is of the first relative degree and this implies the value of the coincidence horizon is equal $H = \rho = 1$. The feedback controller gain vectors $\mathbf{K}_j$, $j = 1, 2$, which are defined in (44) become in the case of selected design parameters:

$$\mathbf{K}_1 = [\,0.5822 \quad -0.2288 \quad -0.4249\,], \tag{56}$$

$$\mathbf{K}_2 = [\,0.4755 \quad -0.1495 \quad -0.3974\,], \tag{57}$$

and the closed-loop system matrices (46) are the following:

$$\begin{aligned}
\mathbf{A}_{c,11} &= \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ 0 & 0 & 0.9200 \end{bmatrix}, \\
\mathbf{A}_{c,12} &= \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ -0.1067 & 0.0793 & 0.9475 \end{bmatrix}, \\
\mathbf{A}_{c,21} &= \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ 0.1067 & -0.0793 & 0.8925 \end{bmatrix}, \\
\mathbf{A}_{c,22} &= \begin{bmatrix} 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \\ 0 & 0 & 0.9200 \end{bmatrix}.
\end{aligned} \tag{58}$$

Note that the index l is dropped in Equations (56)–(58) since the simplified analysis is performed due to the equal output matrices in all fuzzy domains. The equilibrium of the proposed PFCFM system is globally asymptotically stable if there exist a common positive definite matrix $\mathbf{P}$ such that

$$\mathbf{A}^{\mathrm{T}}_{c,ji}\mathbf{P}\mathbf{A}_{c,ji} - \mathbf{P} \prec 0, \quad j = 1, 2,\ i = 1, 2, \tag{59}$$

i.e. a common $\mathbf{P}$ has to exist for all possible subsystems. If we can find a common $\mathbf{P}$ under given assumption, the PFCFM control law *quadratically stabilizes* the system.

The common $\mathbf{P}$ problem can be solved in a way to find a positive definite matrix for each subspace and check if it solves the problem in other subspaces. If we want to prove the stability of the proposed system we have to find a common **P** or determine that no such matrix exists. It can be shown that positive definite symmetric matrix **P** which is equal to

$$\mathbf{P} = \begin{bmatrix} 23.9222 & -2.3365 & -1.1683 \\ -2.3365 & 35.9805 & -7.5937 \\ -1.1683 & -7.5937 & 54.1861 \end{bmatrix} \tag{60}$$

satisfies the conditions given in (59) which means that the PFCFM system in the case of magnetic suspension system is globally asymptotically stable.

The other way is to solve the system of linear matrix inequalities, as defined in (48). In our case the matrix $\mathbf{A}_c$ is defined as

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{A}_{c,11} & 0 & 0 & 0 \\ 0 & \mathbf{A}_{c,12} & 0 & 0 \\ 0 & 0 & \mathbf{A}_{c,21} & 0 \\ 0 & 0 & 0 & \mathbf{A}_{c,22} \end{bmatrix}. \tag{61}$$

A positive definite matrix $\mathbf{P}_c$ have to be find using LMI convex optimization techniques to solve the system of linear matrix inequalities

$$\mathbf{A}_c^{\mathrm{T}}\mathbf{P}_c\mathbf{A}_c - \mathbf{P}_c \prec 0, \tag{62}$$

where $\mathbf{P}_c \succ 0$ equals:

$$\mathbf{P}_c = \begin{bmatrix} \mathbf{P} & 0 & 0 & 0 \\ 0 & \mathbf{P} & 0 & 0 \\ 0 & 0 & \mathbf{P} & 0 \\ 0 & 0 & 0 & \mathbf{P} \end{bmatrix}. \tag{63}$$

In the case of magnetic suspension system controlled by PFCFM we can find a matrix **P** which is symmetric and positive definite and is given next:

$$\mathbf{P} = \begin{bmatrix} 10.48 & -5.17 & -4.69 \\ -5.17 & 32.40 & -28.00 \\ -4.69 & -28.00 & 37.14 \end{bmatrix}. \tag{64}$$

This proves the global asymptotic stability of the proposed predictive control system where the reference trajectory was given as $a_r = 0.92$ and the horizon is $H = 1$. We can prove the global asymptotic stability for the interval $0 < a_r < 0.94$ at coincidence horizon $H = 1$. For the values $0.94 \leqslant a_r < 1$ we cannot find the common positive definite symmetric matrix **P**, and we cannot assure the stability. This does not mean that the system is unstable for those values of parameter a_r since the test only gives sufficient conditions for the stability.

At the end the simulation of the proposed system has been made and the reference, output and control signals are given in Figure 4. The design procedure can be done by defining the coincidence horizon equal to the relative degree of the model and finding the parameter a_r which assures the system stability – in our case $a_r = 0.92$ was used.

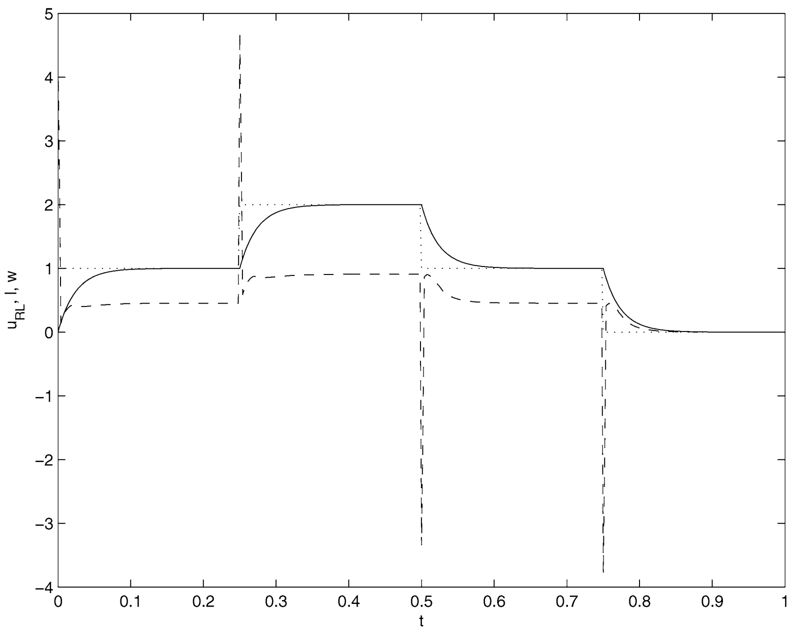

Figure 4. The PFCFM responses: the control signal u_{RL} – dashed, the output signal l – solid, the reference signal w – dotted.

5. Conclusion

The proposed design methodology and stability analysis of parallel distributed fuzzy model-based predictive control give a very powerful framework for nonlinear system control. The idea of blending single control laws for each fuzzy model rule is very simple and natural and has been used in many fuzzy control approaches. The proposed control algorithm is developed in state space domain and is given in analytical form. This is an advantage in comparison to optimization based control schemes especially in the sence of realization in real-time. The problem of fuzzy model based control stability is in the case of parallel distributed fuzzy model-based predictive control reduced to linear matrix inequalities problem which can be solved by convex programming. The design procedure and stability analysis have been shown on magnetic suspension system which is an example of a nonlinear plant.

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